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RYSER'S CONJECTURE FOR TRIPARTITE 3-GRAPHS

RON AHARONI

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We prove that in a tripartite 3-graph $\tau \le 2\nu$.

1. Preliminaries

A *hypergraph* is a set of subsets, called *edges*, of some ground set, whose elements are called *vertices*. A hypergraph is called *r-uniform* (or an *r-graph*) if all its edges are of the same size, r. An r-uniform hypergraph is called *r-partite* if its vertex set $V(H)$ can be partitioned into sets $V_1,\dots,V_r$ (called the "sides" of the hypergraph) in such a way that each edge meets each V_i in precisely one vertex.

A *matching* in a hypergraph is a set of disjoint edges. The *matching number*, $\nu(H)$, of a hypergraph H is the maximal size of a matching in H. Another parameter, introduced in [1], is defined as follows. We say that a set K of edges *pins* another set F of edges if every edge in F is met by some edge from K. The *matching width* of H, denoted by $mw(H)$, is the maximum over all matchings M in H of the minimal size of a set of edges from H pinning M. An easy observation concerning this parameter is:

Lemma 1.1. *In an r-uniform hypergraph H one has $\nu(H) \le r \cdot mw(H)$.*

This is clear, since each matching in H is pinned by $mw(H)$ edges, containing together at most $r \cdot mw(H)$ vertices.

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A *cover* of a hypergraph H is a subset of $V(H)$ meeting all edges of H. The *covering number*, $\tau(H)$, of H is the minimal size of a cover of H. Obviously, $\tau \geq \nu$ for all hypegraphs. In an r-uniform hypergraph $\tau \leq r\nu$, since the union of the edges of a maximal matching forms a cover. The cornerstone of matching theory is König's theorem [5], which states that in bipartite graphs $\tau = \nu$. Ryser's conjecture, made in the early 1970-s, is an extension of this result to r-partite r-uniform hypergraphs:

Conjecture 1.2. In an r-partite r-uniform hypergraph (where $r > 1$), $\tau \leq (r-1)\nu$.

The conjecture appeared in the Ph.D thesis of Henderson, a student of Ryser. Independently, at around the same time, it was conjectured in a stronger form by Lovász [6].

A truncated projective plane (that is, a projective plane from which a vertex is deleted together with all edges incident with it) witnesses the sharpness of the conjecture. The sides in this case are the sets of vertices which, together with the removed vertex, had formed edges in the projective plane. The matching number is 1, while the minimal covers are the sides, which are of size 1 less than the size of the edges (namely, the number of sides of the hypergraph).

There are two directions in which Ryser's conjecture has been studied. The first was fractional versions. Füredi [2] proved that $\tau^* \leq (r-1)\nu$, where τ^* is the fractional covering number (where one is allowed to cover by "fractions" of vertices, i.e by putting weights on vertices). This result is true, in fact, for a wider class of hypergraphs than just r-partite - all hypergraphs not containing a copy of the projective plane. Lovász [6] proved that for r-partite r-graphs $\tau \leq \frac{r}{2}\nu^*$. Another direction was proving bounds for small values of r. In fact only the case $r = 3$ was studied for general ν. The bounds for this case were improved successively: $\tau \leq \frac{25}{9}\nu$ [7], $\tau \leq \frac{8}{3}\nu$ [8], and finally $\tau \leq \frac{5}{2}\nu$ [4].

In the case of bipartite graphs König's theorem is closely related to Hall's theorem [3], and the two are easily derivable from each other. So much so, that they are sometimes called jointly the "König-Hall theorem". In order to pass from one to the other, a bipartite graph is regarded in a special way: one side of it is singled out, and each of its vertices is replaced by a set, consisting of the vertices in the other side to which it is connected. If we do the same in the case of r-partite hypergraphs, each vertex in the side which is singled out represents a set of *edges*, each being of size $r-1$. Thus, while the object which results in the bipartite case is a family of sets (as in Hall's theorem), here we have a family of *hypergraphs*. The notion of a system of distinct representatives is replaced in this case by that of a system of *disjoint* representatives. Its definition is as follows:

Let $\mathcal{A}=\{H_1,\ldots,H_m\}$ be a family of hypergraphs. A *system of disjoint representatives* (abbreviated "SDR") for $\mathcal{A}$ is a function $f:\mathcal{A}\to\bigcup_{i=1}^m H_i$ such that $f(H_i)\in H_i$ for all i and $f(H_i)\cap f(H_j)=\emptyset$ whenever $i\neq j$.

In this paper we prove Ryser's conjecture for the case $r=3$. The main tool in the proof is the following extension of Hall's theorem, proved in [1]:

Theorem 1.3. *If $mw(\bigcup\mathcal{B})\geq|\mathcal{B}|$ for every subfamily $\mathcal{B}$ of $\mathcal{A}$ then $\mathcal{A}$ has a SDR.*

The proof of this result was topological, that is, it used Sperner's lemma.

2. Ryser's conjecture for tripartite 3-graphs

The proof of the main result follows closely the usual proof of König's theorem from Hall's theorem. There, the core of the proof is in a "deficiency" version of Hall's theorem. Likewise, here we shall need a deficiency version of Theorem 1.3.

Definition 2.1. The *deficiency* $def(\mathcal{A})$ of $\mathcal{A}$ is the minimal natural number d such that $mw(\bigcup\mathcal{B})\geq|\mathcal{B}|-d$ for every subfamily $\mathcal{B}$ of $\mathcal{A}$.

Theorem 2.2. *Every family $\mathcal{A}$ of hypergraphs has a partial SDR of size at least $|\mathcal{A}|-def(\mathcal{A})$. That is, there exists a subfamily $\mathcal{D}$ of size at most $def(\mathcal{A})$ of $\mathcal{A}$ such that $\mathcal{A}\setminus\mathcal{D}$ has a SDR.*

Proof Write $d=def(\mathcal{A})$, and let $v_1,\ldots,v_d$ be elements which do not belong to $V(\bigcup\mathcal{A})$. Add to each $H\in\mathcal{A}$ all singletons $\{v_i\}$, namely replace it by the hypergraph $H'=H\cup\{\{v_1\},\ldots,\{v_d\}\}$. Since a sigleton can only be pinned by itself, the matching width increases by this procedure by d. This means that the family $(H':H\in\mathcal{A})$ satisfies the condition of Theorem 1.3, and thus by that theorem has a SDR f. Letting $\mathcal{D}$ be the subfamily of $\mathcal{A}$ consisting of all hypergraphs H such that H' is matched to some $\{v_i\}$, and matching each $H\in\mathcal{A}\setminus\mathcal{D}$ to the edge matched to H' by f, yields the desired partial SDR of $\mathcal{A}$. ∎

We can now prove our main theorem:

Theorem 2.3. *Ryser's conjecture is true for $r=3$. That is, in a tripartite 3-graph $\tau\leq 2\nu$.*

Proof Let Γ be a 3-partite hypegraph with sides V_1, V_2, V_3. We view one side (say, V_1) of Γ as a family $\mathcal{A}$ of hypergraphs, as described above. Let $\mathcal{B}$ be a subfamily of $\mathcal{A}$ at which the deficiency $d = def(\mathcal{A})$ is attained, namely $mw(\bigcup\mathcal{B}) = |\mathcal{B}| - d$.

By Lemma 1.1, we have $\nu(\bigcup\mathcal{B}) \le 2mw(\bigcup\mathcal{B})$. Since the edges in $\bigcup\mathcal{B}$ form a bipartite graph, it follows by König's theorem that $\tau(\bigcup\mathcal{B}) \le 2mw(\bigcup\mathcal{B})$. Let $\mathcal{C} = \mathcal{A} \setminus \mathcal{B}$, and write $|\mathcal{B}| = b, |\mathcal{C}| = c$. Then, by the above, $\tau(\bigcup\mathcal{B}) \le 2mw(\bigcup\mathcal{B}) = 2(b-d)$. Let X be the set of vertices in V_1 corresponding to the elements of $\mathcal{C}$, together with vertices in a minimal cover of $\bigcup\mathcal{B}$. Clearly, X is a cover of Γ, and its size is at most $c + 2(b-d)$. By Theorem 2.2, on the other hand, we have $\nu(\Gamma) \ge |V_1| - d = c + b - d$. Combining all this we have $\tau(\Gamma) \le |X| \le c + 2(b-d) \le 2(c+b-d) \le 2\nu(\Gamma)$. ∎

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Ron Aharoni

Department of Mathematics
Technion, Haifa
Israel 32000
ra@tx.technion.ac.il